Lecture 17: CTMCs

Remark: Often, like in DTMCs, CTMCs are reversible.

This can simplify computation of SD π.

Def: A CTMC is reversible if $\exists$ prob vector π s.t.

$$\pi_i q_{ij} = \pi_j q_{ji} \quad \forall i, j \in S \qquad \text{(DBE)}$$

If π satisfies DBE, sum over j on both sides:

$$\pi_i(-q_i) = \sum_{i \neq j} \pi_i q_{ij} \Leftarrow \pi Q = 0$$

First-Step Analysis for CTMCs (pretty much same as for DTMCs)

- Hitting Probabilities: $\mathbb{P}[\text{I hit A before B}]$
  - ↳ exactly same as DTMC, by considering hitting probs in the jump chain
  - ↳ only different from DTMC when I consider time-dependent quantities

Hitting Times in CTMCs

Fix $A \subset S$. Define $T_A = \min\{t \geq 0 : X_t \in A\}$.

Q/ How do we compute $\mathbb{E}[T_A \mid X_0 = i]$

A/ Just like for DTMCs, define $h(i) = \mathbb{E}[T_A \mid X_0 = i]$

$$h(i) = 0 \quad \forall i \in A$$

For $i \notin A$:

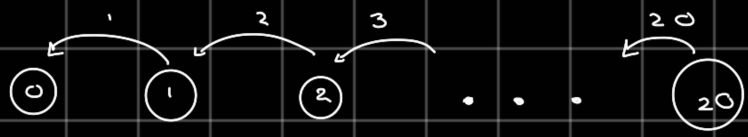
$$h(i) = \mathbb{E}[T_A \mid X_0 = i] = \frac{1}{q_i} + \sum_{j \neq i} P_{ij}\, \mathbb{E}[T_A \mid X_0 = j]$$

$$= \frac{1}{q_i} + \sum_{j \neq i} \frac{q_{ij}}{q_i} h(j)$$

$$\Rightarrow \boxed{\begin{aligned} q_i h(i) &= 1 + \sum_{j \neq i} q_{ij} h(j) && i \notin A \\ h(i) &= 0 && i \in A \end{aligned}}$$

Example: Given 20 lightbulbs, each with iid lifespan $\sim \text{Exp}(1)$.

Q/ How long on avg until all bulbs burn out?



↳ We want to compute $\mathbb{E}[T_A \mid X_0 = 20]$ $\quad A = \{0\}$

$$h(0) = 0$$

$$1h(1) = 1 + 1\cdot 0 \Rightarrow h(1) = 1$$

$$h(2) = \frac{1}{2} + 1h(1) = \frac{3}{2}$$

$$h(3) = \frac{1}{3} + \frac{1}{2} + 1 =$$

$$h(20) = \frac{1}{20} + \frac{1}{19} + \dots + 1 \approx 3.6$$

Uniformization

Motivation: Given CTMC w generator Q, let

$$P_{ij}^t := P\{X_t = j \mid X_0 = i\}$$

Let P^t be the matrix given by $[P^t]_{ij} = P_{ij}^t$

Chapman-Kolmogorov Eqns:

$$P^{t+s} = P^t P^s \qquad t, s \geq 0$$

Claim:

$$P^h = I + hQ + o(h)$$

So: $P^{t+h} \underset{\uparrow \text{Chapman-Kolmogorov}}{=} P^t P^h = P^t(I + hQ + o(h))$

$$\frac{P^{t+h} - P^h}{h} = P^t Q + \frac{o(h)}{h}$$

$$\left.\begin{aligned} &\underset{h \downarrow 0}{\Longrightarrow} \frac{d}{dt} P^t = P^t Q \quad t \geq 0 \\ &P^0 = I \end{aligned}\right\} \text{system of differential equations}$$

$$\underbrace{P^t = e^{tQ} \qquad t \geq 0}_{\text{Kolmogorov Forward Equation}} \Big\} \text{ computing matrix exponentials isn't easy}$$

$$e^{tQ} = \sum_{n \geq 0} \frac{(tQ)^n}{n!}$$

Taylor series

Problem: $(tQ)^n$ is unstable to compute (bc we can have large negative eigenvalues)

"Uniformization" is a technique to avoid the problem and compute transient probs P^t for t small

Assumption: $\exists M$ s.t. $q_i \leq M \quad \forall i \in S$

Construction: Take $\gamma \geq M$ ($\geq q_i \ \forall i \geq q_{ij} \ \forall i,j$)

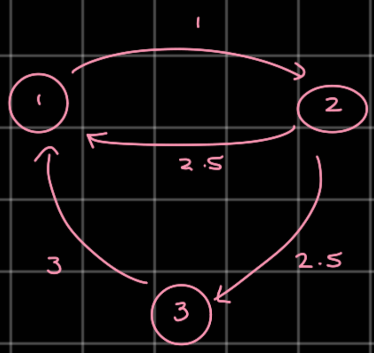
Define a DTMC w/ transitions probs:

$$P_{ij}^{u} := \frac{q_{ij}}{\gamma} \qquad i \neq j$$

$$P_{ii}^{u} := 1 - \frac{q_i}{\gamma}$$

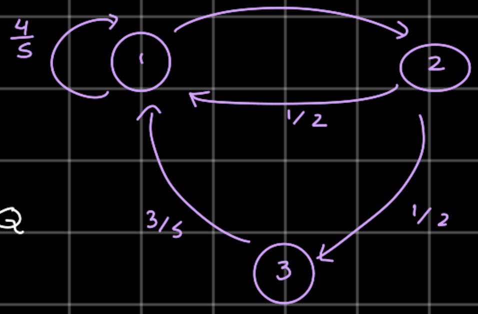
Note: not equal to transition probs for jump chain
⇝ here we have self-loops

Ex: CTMC



$\gamma = 5$

Uniformized DTMC



Note:

$$\underbrace{P_u}_{\text{transition matrix for uniformized chain}} = I + \frac{1}{\gamma} Q$$

$$\Rightarrow Q = \gamma (P_u - I)$$

$$P^t = e^{-tQ} = e^{\gamma t (P_u - I)} = e^{-\gamma t} e^{\gamma t P_u}$$

$$= \underbrace{\sum_{n \geq 0} P_u^n \frac{(\gamma t)^n}{n!} e^{-\gamma t}}_{\text{this is stable to compute}}$$

$$\approx \sum_{n=0}^{k} P_u^n \frac{(\gamma t)^n}{n!} e^{-\gamma t}$$

Random Graphs

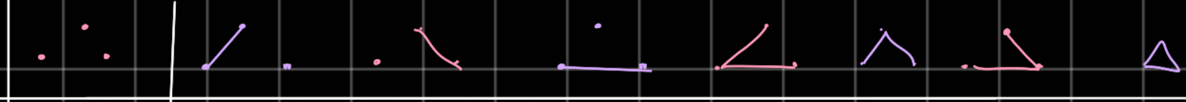
Erdös - Renyi graphs are the "iid coinflip" model in the world of random graphs

Def: Fix $n \geq 1$ and $p \in [0,1]$. A random graph

$G \sim \underbrace{G(n,p)}_{\text{Erdös-Renyi Ensemble}}$ is an undirected graph on n vertices obtained by placing edges w/prob p, independent of all others

Ex: $n=3$

graph G



prob of observing	$(1-p)^3$	$p(1-p)^2$	$p(1-p)^2$	$p(1-p)^2$	$p^2(1-p)$	$p^2(1-p)$	$p^2(1-p)$	p^3

Types of Questions we may want to ask:

- If $n >> 1$, how big/small should p be so that $G \sim G(n,p)$ has property $\mathcal{P}$ with high probability?

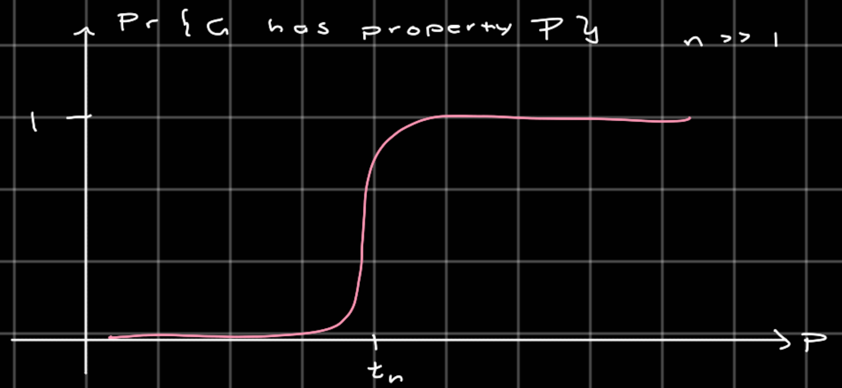
ex: $\mathcal{P}$ = no isolated vertex
$\mathcal{P}$ = graph is connected
$\mathcal{P}$ = graph has a clique of size k

all of these are monotone graph properties (ie, adding edges preserves $\mathcal{P}$.)

Friedgut-Kalai Thm (1996)

Every monotone graph property has a sharp threshold t_n.

ie
- if $p >> t_n \Rightarrow G \sim G(n,p)$ has property $\mathcal{P}$ with high probability (whp) in n.
- if $p << t_n \Rightarrow G \sim G(n,p)$ will not have property $\mathcal{P}$ whp in n



Examples:

① $\mathcal{P}$ = graph has at least 1 edge

→ If $p \ll \frac{1}{n^2}$ then $G \sim G(n,p)$ has no edges whp

→ If $p \gg \frac{1}{n^2}$ then $G \sim G(n,p)$ has an edge whp

② $\mathcal{P}$ = graph contains a cycle.

→ $t_n = \frac{1}{n}$

③ $\mathcal{P}$ = graph contains a "giant component" of size $O(n)$

→ $t_n = \frac{1}{n}$

④ $\mathcal{P}$ = graph is connected

→ $t_n = \frac{1}{n} \log n$